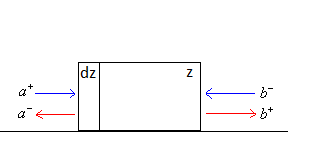
**Muttalib Summary**

A different apprroach, closer in spirit to the derivation of the original DMPK equation, was attempted by Muttalib in 1999. Here we go back to the polar representation of the transfer matrix and revisit the evolution equation of pz(u,υ,λ).



This time we cannot use the maximum entropy ansatz to calculate the pdz(u,υ,λ). Instead a minimal assumption is made that whatever it is, it can be considered independent of u. This could be justified via the argument that in the small dz limit the product uʹυʹ = 1, and so only one of these is a true degree of freedom [but this equality would have to hold to order dz to justify this maneuver]. We will also presume a random phase approximation – namely that pdz will average to zero any combinations of uʹ, υʹ that aren’t conjugate pairs. Ex post facto, we will also stipulate that whatever the distribution pdz(υ,λ) is, it will enforce the following averages – the only ones we will need to compute later on:



where μ1 and μ2 are as yet to be determined phenomenological parameters (μ1 = μ2 = 1 corresponds to max entropy averages). Having specified pdz to our necessary level of specificity, we may develop the perturbative equation for pz(u,υ,λ). But again, we will find that our model allows us to bypass the expansions of u and υ, and write down an equation for pz(λ) directly. To wit, with the first supposition on pdz, we may take the convolution equation, integrate both sides w/r to dμ(u)dμ(υ), and write:



We can divide out the g( ) term in pz and perform the integration over dμ(uʹ), thanks to its measure invariance. Likewise, we can divide by f( ) and perform the dμ(u) integration. Finally, we may divide υ by υʹ† and perform the dμ(υ) integral. This leaves us with:



Now expanding λʹʹ about λ to order dz, etc., we arrive at the following result:



[ξ = (N+1)ℓ/2] But this equation as written is flawed in that it doesn’t conserve probability, i.e., Pz(λ) = J(λ)pz(λ) cannot be written as a total derivative, and no choice of μ1 or μ2, besides the max entropy choice, will remedy this. To surmount this difficulty, the measure J(λ) is ad-hoc renormalized to force probability conservation. Multiplying through by the apriori unknown new measure, J´(λ), and enforcing probability conservation, requires:



With these conditions we can write, with the understanding that we have redefined Pz(λ) = J´(λ)pz(λ),



[ξ is also renormalized to (N+1)ℓ/2μ2]. In this form one can plainly see the DMPK equation is recoverd when γ = 1. γ, then, emerges as the order parameter controlling the transition from the delocalized phase (γ = 1) to the localized phase (γ = 0). There are a few shortcomings here, still, which were redressed by two papers in 2002 and 2014. We’ll elide some of the differences in the two papers so as to present a smoother narrative. First, one would like to put the pdz assumptions on a firmer basis. Secondly, the phenomenological parameters μ1, μ2, were assumed to have no index dependence, which, while the simplest assumption, and consonant ultimately with the one-parameter scaling, isn’t on a sound physical basis. Third, in order to preserve probability conservation, the measure had to be renormalized. And fourth, the equation predicts correlations between the λ’s will go as |λi – λj|γ, when in fact it should go as |λi – λj|1, regardless of the sample’s phase (β = 1).

As for pdz, we keep the random phase approximation, also the presumption that pdz, though not independent of uʹ per seʹ, does enforce uʹυʹ = 1 to O(dz), and then finally supplement these with the following statistics:



taken from Mello paper and Chalker paper respectively. The latter models anisotropy of the reflection amplitude. Writing down the convolution equation again, and aiming to directly compute pz(λ), we integrate it over ∫dμ(u)∫dμ(υ). It turns out that we may perform the ∫dμ(u) integration, owing to measure invariance, just like before. But we are unable to perform the ∫dμ(υ) integration owing to the formal dependence of pdz on uʹ. So the best we can do is:



Next, making reference to Appendix E, we will expand both υʹʹ and λʹʹ about their unperturbed values υ and λ (assuming that we need only go out to second order, as will be true given our model for pdz). Note that formally, one ought to do an expansion in, and then integration over, both υab *and* υab\*, but one can obviate this by employing the υυ† = 1 constraint that relates the two variables. All total the assumptions in this model are:



Next we integrate by parts to take derivatives w/r to υab off of pz(υʹʹ,λ’ʹ). And finally, we will assume that the υ’s all occur in combinations whose fluctuations about their averages can approximately be ignored, and replace them with their mean values. Our result is:



It will turn out that the eigenvector perturbations in the < >Mʹ only involve υ in the following form:



Beyond the supposition that fluctuations about their mean are negligible *enough*, we make a few assumptions regarding these elements. First we presume a random phase approximation whereby this will give us:



where Cmn,mʹnʹ is a function that simply enforces {m,n} = {mʹnʹ}. Since this equation now involves an unknown Kmn(λ), it must be supplemented by an evolution equation for Kmn(λ) itself. To that end, we form Mʹʹ = MMʹ and construct an expression for the new eigenvectors at length z + dz. We find:



And then proceed to evaluate:



Interestingly, this equation will invoke υ in the same combination as before. This time we will encounter products of kmnmʹnʹ. To handle these we make the assumption that the set of indices of each k must separately equate:



This does go slightly beyond a bare random phase assumption, but the class of cases captured is a factor N2 larger than the rest, so we expect this approximation to be innocuous in the large channel liimt. After lengthy algebra we arrive at a generalization of the DMPK equation similar to the prior one, though which does restore the original measure, and promotes μ1 and μ2 to index (and λ-dependent) quantities. We find, for Pz(λ) = J(λ)pz(λ),



Note the version of this equation given in the paper had already specialized to the case where Kmn was λ-independent. Here we present the most general version of the equation. And we find the evolution of Kmn(z) is found to be given by:



where,



So we arrive a result similar in form to Shukla’s version (with his γ = 0), but without an explicit formula for the matrix Kmn. Still, a few properties of interest emerge from the Kmn evolution equation. First, under the assumption that we may use a mean field approximation that Labαβ = KabKαβ (the circumstances under which this might be generally valid are discussed in the paper, but it is also consistent with our prior stated assumption that the probability distribution of Kmn(λ) is nearly a δ function), the diagonal component can be written as:



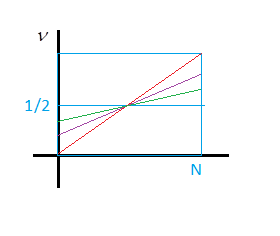
And from this it follows that the only two asymptotically stable eigenvector correlations, in the long length limit are γmn = 0, 1, the latter being the only stable solution, and corresponding to an isotropic dispersion law which returns us to the DMPK equation, consistent with the discussion above in ‘what to expect’. Furthermore, in the small eigenvalue separation limit, the denominator becomes singular, but the finiteness of the Kmm requires γmn → 1, thus preserving the known eigenvalue correlations in the small separation limit [cite Markos paper]. Finally the discussion is rounded out with an equation for the Lyapunov exponents, first produced by many authors prior.



In large and separated λ limit, this approximately reduces to:



which indicates ν­min = νN ≈ (1/2)KNN, and establishes, as has been noted before, a connection between the minimum Lyapunov exponent and the diagonal part of the K-matrix. It is of note that the above (exact) equation predicts the exponents must always sum to N/2, which together with our expecation of the behavior of νmin suggests the following disorder-dependent distribution:



It can be argued that the statistical model for pdz [random phase assumption, uʹυʹ → 1 identity, and statistical correlations in those equations] is not as well motivated as one would prefer. Later unpublished calculations have put these results on a more solid footing. Three different microscopic models were considered: Chalker’s real space model of N conducting parallel 1D chains,

Mello’s model,



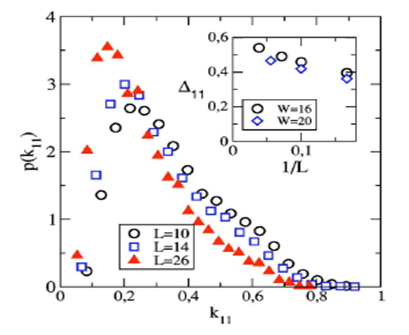
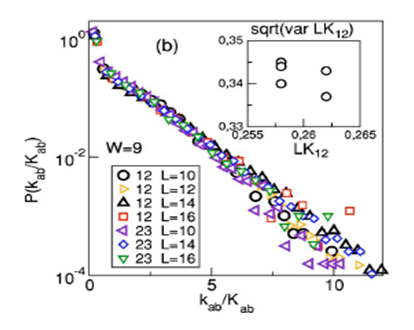
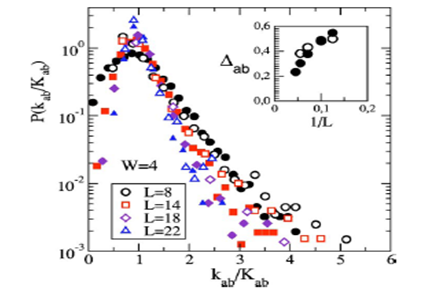
and the white noise model:



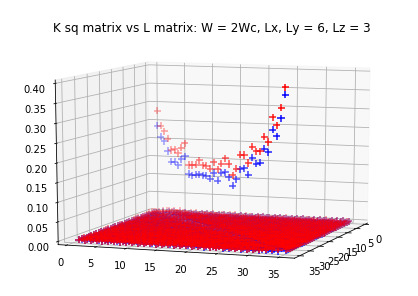
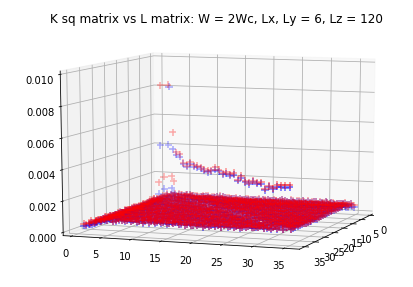
. For each, equation [in Appendix] was used to directly calculate Pz(λ), and equation [in Appendix] was used to calculate the evolution of Kmn(λ). In the first, all eigenvectors occur solely in terms of the previously defined kmnmʹnʹ. Assuming <kmnmʹnʹ> = <kmnmn> Cmnmʹnʹ, and defining Kmn = <kmnmn> as before, one obtains the same GDMPK equation, same evolution equation of Kmn(λ), same equation for νn, and same sum rule. In Mello’s model, all eigenvector combinations occurred solely in the form kmnmʹnʹ = υmʹαυnʹβυmαʹ\*υnβʹ\*σαβCαβαʹβʹ. Again, it was assumed <kmnmʹnʹ> = <kmnmn> Cmnmʹnʹ, and the definition Kmn = <kmnmn> was made. In these terms, the same GDMPK equation, Kmn(λ) evolution equation, and Lyapunov exponents equation + sum rule was obtained. It’s worth mentioning, in this context, that it is easy to see how the equivalent channel model reproduces the DMPK equation, as setting σab = (1+δab)/(N+1)ℓ automatically makes Kmn equal the same and hence γmn to be 1. Lastly, the white noise model reproduced all the results of Mello’s model. It is an open question precisely what features the pdz statistical model needs to have.



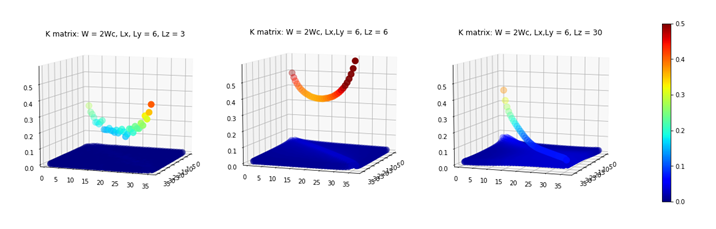
Going back to the GDMPK, it is also an open question whether the assumptions underpinning the replacement of kmnmn ≡ kmn with its average Kmn are warranted, and also whether the matrix γmn does indeed govern the transition in some sense. A numerical analysis of the k-matrix was undertaken in the 2005 paper. The sample graphs below plot its probability distribution in the metallic, critical, and insulating phases.



The distribution is self-averaging (i.e. √var(kab)/<kab> → 0) in the large L (= Lx = Ly = Lz) limit in the metallic phase, justifying our presumption of replacing kab with its average. Though not self averaging, it is highly peaked (note logarithmic scale) at the critical point. It is also not self-averaging in the insulating regime, though it becomes more peaked with increasing L. Still, the lack of self-averaging in these latter two states places our presumption on more tenuous grounds. From another vantage point, if kmnmn is self averaging, then for instance we ought to be able to legitimately replace Lmnmn with Kmn2, as was explicitly done when we argued that γmn → 1 regardless of disorder, if the eigenvalues λm and λn make an anomonously close approach. The graphs below show L and K2 side by side for two different geometries, in the insulating state.

The similitude of the two provides support for treating K as a deterministic variable, to first order approximation. The next question to address is whether the matrix K has any relevance, i.e., whether it is sensitive to the phase of the metal. Plots below show how the matrix K changes shape between the Q1D to cubic geometry in the insulating state. As Lz increases with LT fixed, the distribution of values regresses to the equivalent channels model fairly quickly, and so numerical results are consistent with regression to Q1D behavior, even for strong disorder.



As for K’s ability to capture the transition, consider the following plot of γ12 = 2K12/K11 vs. disorder, W, for various fixed dimensions:



As one can see, in the large L limit, the off-diagonal element γ12 has a fixed value of approximately ¼ at critical disorder, and otherwise asymptotes to 1 in the metallic phase, or 0 in the insulating phase (this doesn’t directly contradict the dK/dz equation since it implicitly keeps the width of the sample fixed). These encouraging results motivated an analytical study of the GDMPK equation, especially in the insulating regime. This was undertaken in a paper in 2005, as well as two in 2009, 10. A drastically simplified non λ-dependent insulating regime (LT >> ξ) model for the matrix Kmn was chosen, based on numerical studies of the matrix elements:



Then it follows γm≠n ≈ γ12 = ξ/8LT. The details we’ll relegate to those papers, but the gist is as follows. Following Beenakers paper, we make the following substitution (λ = sinh2x):



which reduces the FP equation to a Schrodinger equation in imaginary time,



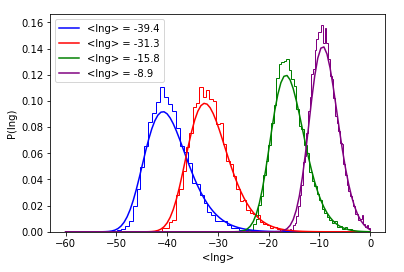
obeying δ-function initial conditions (I think z must really be z/ℓ). Note that only for an index-independent diagonal Kmm do we find this Hamiltonian reduces to a solely two body interaction; otherwise we find additional three body interactions, as Shukla also finds. Since γ is small in the insulating regime, a perturbative expansion for ψ is developed, and partially re-summed to factor out its non-analytic behavior, caused by the singularities in the potentials at x = 0, and xa = xb. The lowest order result is:

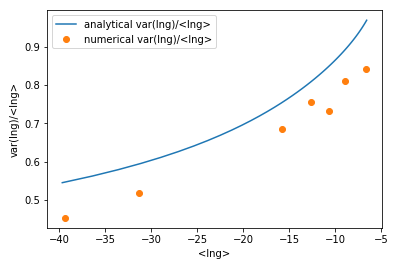
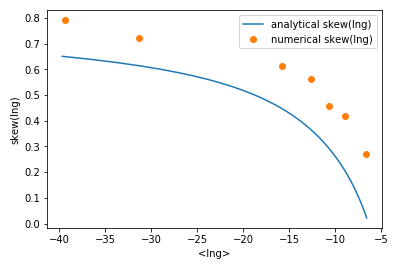


(yeah, if Kz is unitless, then z must be really z/ℓ) From here, the probability distribution of the smallest of these parameters, P(x1,z) is obtained by integrating P(**x**,z) over the N-1 largest xn’s. Since γ12 is small, the repulsion between nearest neighbor x’s is small as well (and hence invalidates the usual Q1D approximation xn+1 >> xn); so these N-1 integrals are performed via a continuum approximation. This gives us a relatively simple analytic formula for P(x1,z). Now x1 provides the dominant contribution to the conductance, and while the eigenvalues are close together owing to their weak repulsion, we still hope to approximate x1 = (1/2)ln(4/g), coming to the following expression for the unnormalized Pz(lng):



where erfc( ) is the complimentary error function. We’ll note that Pz(lng) is governed by two independent parameters: LT/ξ, and z/ξ, which become a single parameter for a cubic geometry. Basically, the first term in f( ) is the Q1D result which presumes the x’s are well separated and effectively independent, as they are when ξ > LT. The second term in f( ) provides an increasingly non-negligible correction as ξ < LT, and stems from the x’s actual interdependence. Typical results for cubes in the deep insulating regime are shown below:



The histogram is taken from numerical simulations [cite Markos]. The solid curves come equation [whatever] in reference [one of those two papers because you need the actual equation to get good matching], with ξ chosen to match mean conductance with the numerical data. Below we have plotted the ratio of var(lng) and <lng> vs. <lng> itself, and then to the right skew(lng) vs. <lng> itself. Points are taken from numerical simulations and the curves from the aforementioned formula. As one can see a the 3D insulator is markedly different from a Q1D insulator. In Q1D var(lng)/<lng> is *constant*, rather than decreasing. And the skewness of P(lng) should also decrease to zero, rather than *increase*, though this seems to indicate that the ln-normal distribution is a better approximation as one gets closer to the critical point. So as one can see, even under gross simplification (and gross approximate solution of) the GDMPK equation does a much better job modelling a real metal. Even still, there are problems. The expression for P(g) predicts that in the far asymptotic regime var(lng) ~ <ln(1/g)>/ln<ln(1/g)>, which doesn’t match current numerical conjectures, though it does fit the data just as well. It also overestimates the variance, and relatedly, underestimates the skew. But this level of verisimilitude is perhaps not to be expected given the number of approximations which had to be made to get to the analytical result. More importantly, the localization length, ξ, does not emerge naturally from the theory, but must rather be used as a fitting parameter [can Markos give estimates of ξ to see how close these match?]. Numerical studies indicate that the matrix γmn does indeed capture the transition, *but* this fact does not emerge from the theory itself, i.e., there is no present analytical indication that γ ≈ ¼ is a critical value in any sense. Perhaps a more extensive study of the GDMPK equation and its Kmn(z) counterpart would redress this. It would also be advisable to check the alleged sum rule. If this breaks down in the insulating regime, then this would be a clear indication of the limits of this perturbative approachx. It is possible that the present theory does a better job modeling a highly disordered conductor, but doesn’t yet describe a true insulator. On a side note, alternative approaches have been explored, such as examining the possibility of writing a closed equation for the probability distribution of traces, ρk, as Mello has done. But the evolution of these traces likewise fails to close for these non-equivalent channels models of pdz.

In Beenaker’s review, pg. 72, he lists references for papers that examine the generalization of DMPK to higher dimensions.